

Anisotropic singularities and modified gravity

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Abstract

In four space-time dimensions, there exists a special infinite-parameter family of chiral modified gravity theories. All these theories describe just two propagating polarizations of the graviton. General Relativity with an arbitrary cosmological constant is the only parity-invariant member of this family. Modifications of General Relativity can be arranged so as to become important in regions with large Weyl curvature. We review how these modified gravity theories arise within the framework of pure-connection formulation. We introduce a new parametrisation of this family of theories that, apart from the fundamental connection field, uses certain set of auxiliary fields. We show how the Kasner singularity of General Relativity is resolved in a particular modified gravity theory of this type. There arises a new asymptotically De Sitter region “behind” the would-be singularity, the complete solution thus being of a bounce type. Although the effective metric based on this solution still contains singularities and experiences changes of signature, the fundamental connection field is everywhere regular.

1 Introduction

The aim of this work is to study the effects of a particular modification of General Relativity (GR), see below, on the Kasner singularity of GR. The Kasner behaviour is realised in an evolving spatially homogeneous but anisotropic Universe. The simplest situation is the Bianchi I cosmology, with the spatial metric being independent of the Cartesian spatial

coordinates. A more rich Bianchi IX case realises the Kasner behaviour between the bounces of the “cosmological billiard” [1]. The Kasner behaviour is widely expected to capture the essence of a generic spacelike singularity of General Relativity.

It is generally believed that the sought theory of quantum gravity will “resolve” the singularities of classical GR, or at least describe the physics in regions where curvature becomes Planckian. Physically interesting singularities are of spacelike type; such is the singularity inside the Schwarzschild black hole or the Kasner singularity to be considered in this paper. However, the existing candidate quantum theories of gravity are not (yet) capable of dealing with the physically relevant anisotropic spacelike singularities.

In this situation, one can hope that some information about the fate of singularities in the future quantum gravity could be obtained by studying classical theories gravity modified in the domain of high curvatures. However, one immediately faces the following problem: In four spacetime dimensions, there is essentially a unique dynamical theory of spacetime metrics with second-order field equations, which is GR.

General relativity theory is commonly regarded as just the low-energy part of some more general effective theory of gravity with the Lagrangian containing higher powers of curvature. Unfortunately, any such theory with a finite number of higher curvature terms leads to higher-order partial differential field equations. This means that such a theory propagates more degrees of freedom compared to GR. If one is not willing to change the dynamical character of the theory, one has to consider Lagrangians with an infinite number of higher curvature terms, together with some principle that controls this infinite series. However, availability of such a principle to a large extent requires the availability of the theory of quantum gravity. This discussion illustrates how hard it is to modify gravity while keeping its dynamical content (spin-two massless degrees of freedom) intact.

At the same time, one can modify GR while preserving the second-order character of field equations as follows. It is known that GR can be reformulated using first-order formalism, in which one introduces an additional connection field. The field equation for this connection is then the statement that it is compatible with the metric. If one eliminates the connection by solving this (algebraic) field equation, one returns to the familiar second-order formulation of GR. However, in the presence of a non-zero cosmological constant, one can first solve the field equation for the metric. This fixes the metric to be a particular function of the curvature of the connection. Inserting this solution for the metric back into the Lagrangian, one obtains a second-order theory with the connection as the only dynamical variable. Such descriptions of GR are known for a long time, historically being first proposed by Eddington and later studied by Schrödinger. Similar other descriptions exist, depending on the type of connection; see below. Collectively, they can be called “pure-connection” formulations of GR.

When connection is used as the main variable, a particular function of the curvature of this connection serves as a Lagrangian density. The resulting field equations are then those of GR in the pure-connection formulation. When the connection satisfies its Euler–Lagrange equations, the metric constructed in a particular way from the curvature satisfies the Einstein equations. It is here that the pure-connection formulations open a new possibility of modification. One can consider modified functions of the curvature as Lagrangians

(if these exist). Such modified theories will still have second-order field equations. In some cases, they will have the same number of degrees of freedom as in GR. Furthermore, such modified theories of gravity may have the potential to implement a desirable property of modifications of GR — involving higher powers of curvature while at the same time keeping the order of field equations intact.

While these ideas may seem promising, the modifications they entail in most cases change the dynamical character of the theory and introduce more propagating degrees of freedom. One of the most convincing ways to argue this is via the so-called on-shell methods, which is also one of the most economical ways of proving the uniqueness of GR; see [2]. By the method of on-shell scattering amplitudes, one finds that it is to a large extent irrelevant which field is used to describe gravitons. There are only a very limited number of possibilities for the graviton interactions. If one starts with a non-chiral (see below) description of gravitons, one finds GR to be the unique theory of interacting massless spin-two particles with second-order field equations.

The only new possibility arises if one considers chiral description of gravitons; see [3]. One then finds that there is an infinite-parametric family of modified gravity theories with second-order field equations. Importantly, these are all chiral theories (apart from GR, which resides in this family of theories as the only parity-invariant member). As a consequence, these modified gravity theories, strictly speaking, exist only as those of complexified metrics (or as those of Riemannian or split-signature metrics).

Thus, if one considers complexified four-dimensional metrics (alternatively, if one works with Euclidean or split signature), then GR is no longer the only possible gravity theory with second-order field equations. For such modified gravity theories, in general, no reality condition is known that selects a half-dimensional slice in the space of complex-valued metrics and allows for a physical interpretation of the theory. For this reason, modified chiral four-dimensional gravity theories at present exist, strictly speaking, only in the mathematical sense. However, in some special situations, e.g., for field configurations with a large number of symmetries, the sought-for real slice is straightforwardly identified. In these situations, one can speculate about the physical effects of the modification.

One such example was given some time ago in [4, 5] (see also [6]), where the spherically symmetric problem was analysed. It was found that there is a peculiar mechanism that resolves the singularity inside the black hole in terms of the fundamental connection fields. In a related work [7], the effects of modifications on the theory of cosmological perturbations were under investigation. The goal of the present paper is to study the fate of the Kasner space-like singularity of GR in modified gravity theories of this type.

Previous works on the physics of the modified gravity theories under consideration used a first-order formulation. The corresponding computations involved many auxiliary fields and were quite cumbersome. In this paper, we employ a much more economical pure-connection formulation, which simplifies the analysis tremendously.

Apart from direct application to Bianchi I models, this paper develops a new parametrisation of the underlying modified gravity theories. The new description uses not just a connection, but also a number of auxiliary fields assembled into a 3×3 matrix. We refer to a description with auxiliary fields as “mixed” in the main text, because it is half-way between

the first-order formulation of the theory with even more fields, and the pure-connection formulation. The new mixed description turns to be quite powerful. In particular, it allows for a description of GR with zero cosmological constant in an essentially pure-connection setting, something which is impossible if one works just with connections.

For concreteness, we concentrate in this paper only on the simplest but already quite interesting setting of a (flat) spatially homogeneous but anisotropic universe. In the metric context, this is known as the Bianchi I model. Our main result is that a generic class of modified gravity theories of the type studied here resolves the Kasner singularity of the GR solution of this model. This resolution occurs in a way similar to the case of black hole discussed above: although the effective metric based on this solution still contains singularities and experiences changes of signature, the fundamental connection field is everywhere regular.

This paper is organised as follows. In Section 2, we describe the known pure-connection formulations of gravity. To obtain the chiral pure-connection formulation we start with the so-called Plebanski first-order description. We then show how the pure-connection formulation of GR arises, and motivate the modified theory as a certain generalisation of the chiral pure-connection formulation of GR. In this section, we also present the “mixed” new parametrisation of the modified theory. In Section 3, we specialise to the sector of interest, which is that of Bianchi I connections. Here we introduce the evolution equations for the connection components and obtain the metric described by this connection. In Section 4, a convenient choice of the time variable is made, which allows us to solve the evolution equations for our theories in full generality, without making any assumption about the function that controls the modification. Here we also establish some general properties of the solution. Section 5 specialises to the case of GR with non-zero cosmological constant. We describe the connection components in this case, and obtain the Kasner behaviour of the metric, which is realised near the singularity. In Section 6, we analyse the solution in the case of a particular one-parameter family of modifications. It is here that we will see how the modification resolves the Kasner singularity. We end up with a discussion.

There is a number of appendices at the end of the paper. In the first Appendix, we start with a more general assumption about the connections of interest, and derive the ansatz used in the main text. In the second Appendix, for completeness, we derive the GR solution working in physical (proper) time. This is possible, but is considerably more involved than the derivation presented in the main text. In the last Appendix, we derive the Bianchi I solution in GR with $\Lambda = 0$ working in the “mixed” parametrisation with auxiliary fields.

2 The pure-connection formulation(s) of gravity

As we have already described in the Introduction, in the gauge-theoretic approach to gravity one takes connection, not metric, as the main variable. The connection then plays the role of the potential for the metric, schematically $g = \partial A$, so that the metric is constructed from the first derivative of the potential.

In the pure-connection formulation, GR can be straightforwardly modified by considering an arbitrary function of the curvature as the Lagrangian. This is guaranteed not to change the second-order character of field equations. Sometimes it also does not change the number of propagating degrees of freedom of the theory; see below.

At the very basic level, the way this approach works is as follows. There is some action principle with the only field appearing in the Lagrangian being the connection A . The arising Euler–Lagrange equations are second order partial differential equations for the connection, which schematically can be expressed as $\partial^2 A = A$. For an arbitrary connection, not necessarily satisfying its field equations, one can construct a certain metric g_A from (the first derivatives of) A . The Ricci tensor of g_A is then of third order in derivatives of A . By using the connection field equations, $\partial^2 A = A$, the Ricci tensor gets converted into an object of first order in derivatives of A . For the case of GR in this framework, this is just a multiple of the metric itself, and we obtain the vacuum Einstein equations. Thus, schematically,

$$\text{Ricci}[g_A] = \partial^2 g_A = \partial^3 A \Big|_{\text{on-shell}} = \partial A = g_A. \quad (1)$$

This explains how the field equations for the connection imply the Einstein equations for the metric g_A . Another similar line of identities can be written for the Weyl curvature of the metric g_A . In this case, the connection field equations imply that this Weyl curvature becomes (on-shell) identical to the curvature $F[A]$ of the connection itself:

$$\text{Weyl}[g_A] = \partial^2 g_A = \partial^3 A \Big|_{\text{on-shell}} = \partial A = F[A]. \quad (2)$$

The on-shell identification of the Weyl curvature with the curvature of A makes it clear that an interesting class of modifications of gravity can be obtained in this formulation by simply appending the Lagrangian of GR with various scalars constructed from $F[A]$. These modifications will become important in regions where Weyl curvature is large, which is what is desired for the problem of anisotropic singularities. At the same time, such modifications of the Lagrangian for gravity do not lead to an increase in the order of field differential equations, which remain to be second-order partial differential equations for A .

The above discussion makes it clear that this approach to modifying GR is available in any “pure-connection” formulation. There are several possible such formulations, as we shall now review. However, the above argument only guarantees that the order of field differential equations is unchanged, but it does not guarantee that the number of propagating degrees of freedom remains the same after modification. And indeed, the argument with scattering amplitudes [3] suggests that most of such modifications will introduce additional degrees of freedom.

After a brief review of the known pure-connection formulations, we will focus on the chiral pure-connection formulation, which is capable of modifying gravity without introducing new degrees of freedom.

2.1 Eddington–Schrödinger formulation

This is the pure-connection formulation of GR that is as old as the Einstein–Hilbert theory itself. As with all pure-connection formulations, a particularly simple way to get it is to start with the first-order formulation in which one has both the connection and the metric as independent variables. In the case of GR, this is the Palatini formulation, with the affine connection $\Gamma_{\mu\nu}^{\rho}$ and the metric $g_{\mu\nu}$. One then “integrates out” the metric $g_{\mu\nu}$, i.e., solves the field equations for g and substitutes the solution back into the action. This is only possible to do in the presence of a non-zero cosmological constant. As a result, an action principle is obtained that contains only $\Gamma_{\mu\nu}^{\rho}$. In the Eddington–Schrödinger case it is constructed from the (symmetric part of the) Ricci tensor

$$R_{\mu\nu}[\Gamma] := R_{(\mu\alpha\nu)}^{\alpha}[\Gamma], \quad (3)$$

where $R_{\mu\nu\rho}^{\sigma}[\Gamma]$ is the curvature of $\Gamma_{\mu\nu}^{\rho}$, the latter being assumed to be symmetric in its lower two indices. The pure-connection action principle is then

$$S[\Gamma] = \frac{1}{8\pi G\Lambda} \int \sqrt{\det(R_{\mu\nu}[\Gamma])} d^4x. \quad (4)$$

It is not hard to check that the square root of the determinant of $R_{\mu\nu}[\Gamma]$ has the correct density weight so that the integral over the spacetime is well-defined.

The field equations that result from (4) are

$$\nabla_{\rho} R^{\mu\nu}[\Gamma] = 0, \quad (5)$$

where $R^{\mu\nu}[\Gamma]$ is the inverse of $R_{\mu\nu}[\Gamma]$. If one now makes a definition

$$g_{\Gamma} := \frac{1}{\Lambda} R_{\mu\nu}[\Gamma], \quad (6)$$

then equation (5) tells that $\Gamma_{\mu\nu}^{\rho}$ is the g_{Γ} -compatible connection. The definition (6) of g_{Γ} is then the vacuum Einstein equation.

The above description shows that this pure-connection formulation only makes sense when $\Lambda \neq 0$. Another feature of this formulation is that the field equations (5) are highly non-polynomial in the derivatives of Γ , containing the inverse of $R_{\mu\nu}[\Gamma]$. Both these features are shared by any pure-connection description of gravity.

Theory (4) can be modified by considering more general functionals of Γ . For example, one can imagine dropping the symmetrization in (3) while keeping the same action (4). One can also take another contraction $R_{\mu\nu\alpha}^{\alpha}$ and form the same Eddington functional from it. More generally, any scalar density of weight one constructed from the Riemann tensor $R_{\mu\nu\rho}^{\sigma}(\Gamma)$ can serve as Lagrangian. It would be interesting to classify the freedom available here, and also to study the arising modifications. As we have already indicated above, it should be expected that modifications of the Eddington formalism will describe more degrees of freedom compared to those in GR.

2.2 The spin-connection formulation

To get this formulation, one proceeds in a similar way starting from a different first-order description. We can take it to be the Einstein–Cartan tetrad formulation. The trick of integrating out the vielbein from the first-order formulation can be carried out in $(2 + 1)$ dimensions [10]. This is best explained in Section 3.4 of [11]. Again, the nonzero cosmological constant is essential, and again one obtains a Lagrangian built from the square root of the determinant of the matrix of curvatures, which is characteristic of these approaches.

We are not aware of any pure spin-connection formulation in four spacetime dimensions. To obtain such a formulation, one would need to integrate out the tetrad, which in this case amounts to solving the equation

$$\epsilon^{IJKL}\theta^J \wedge F^{KL}[\omega] = \Lambda \epsilon^{IJKL}\theta^J \wedge \theta^K \wedge \theta^L, \quad (7)$$

where ω^{IJ} is the spin connection, θ^I is the tetrad, and the capital indices are the “internal” four-dimensional ones. One would need to solve the above equation for the tetrad to obtain an algebraic function of the curvature $F_{\mu\nu}^{IJ}$. The pure-connection Lagrangian is then the determinant of the resulting tetrad. In contrast to all cases already considered where the solution presents itself readily, equation (7) is very non-trivial to solve, although it is not impossible that the corresponding pure-connection formulation does exist. It would be interesting to find it.

2.3 The chiral Plebanski formulation

In the chiral approach, one starts with the so-called Plebanski formulation of GR [12]; see also [13] and [14]. This is a first-order formulation, with an $\mathfrak{su}(2)$ Lie-algebra-valued two-form field and a connection as independent variables. Let us start by reviewing this formulation, which is going to be important for the present paper.

We denote the Lie-algebra indices by lower-case Latin letters $i, j, k, \dots = 1, 2, 3$. The basic fields are a Lie-algebra-valued two-form field with components B^i and a connection one-form with components A^i . There is also a Lagrange multiplier field Ψ^{ij} , which is symmetric and traceless. The action of the theory is

$$S[B, A, \Psi] = i \int B^i \wedge F^i - \frac{1}{2} \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) B^i \wedge B^j. \quad (8)$$

Here, Λ is the cosmological constant, which may be zero. The imaginary unit $i = \sqrt{-1}$ in front of the action is needed in order to make it real for fields satisfying the reality conditions as appropriate for Lorentzian signature; see below. One would not need this pre-factor in either Riemannian or split signature.

Let us consider the field equations stemming from (8). First, varying with respect to B^i , we get

$$F^i = \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) B^j. \quad (9)$$

The Euler–Lagrange equation for the connection is

$$d_A B^i = 0. \quad (10)$$

Finally, there is the equation obtained by varying with respect to Ψ^{ij} :

$$B^i \wedge B^j \sim \delta^{ij}. \quad (11)$$

This equation can be understood as telling that B^i “come from a tetrad” in the sense that B^i satisfying this equation contain no more information than that provided by the metric plus a choice of an $\text{SO}(3)$ frame at every spacetime point. Equation (10) can be solved for A^i in terms of derivatives of B^i whenever B^i are non-degenerate (that is, the three two-forms B^i are linearly independent). In particular, equation (10) can be solved if the two-forms B^i satisfy (11), in which case the solution A^i can be shown to be just the self-dual part of the Levi-Civita connection for the metric described by B^i . Equation (9) then becomes a statement that the curvature of the self-dual part of the Levi-Civita connection of a metric is self-dual as a two-form. This is known to be equivalent to the Einstein condition, which shows that (8) is indeed a description of GR. When all field equations are satisfied, i.e., on-shell, the field Ψ^{ij} is identified with the self-dual part of the Weyl curvature. For more details on this formulation, the reader is referred, e.g., to [14].

2.4 The metric

In the above description, we have mentioned the fact that B^i satisfying (11) contain no more information than that available in the metric (up to gauge rotations). This metric is determined by the two-form fields directly, as we now review.

The main point is the geometric idea that, in four dimensions, one knows the conformal class of the metric (i.e., the metric modulo multiplication by an arbitrary function) if one knows which two-forms are self-dual. Therefore, by declaring the triple of two-forms B^i to be self-dual, one fixes the conformal class of the metric uniquely. Explicitly, the conformal class is given by the following representative due to Urbantke [15]:

$$g_{\mu\nu} \sqrt{\det g} \sim \tilde{\epsilon}^{\alpha\beta\gamma\delta} \epsilon^{ijk} B_{\mu\alpha}^i B_{\nu\beta}^j B_{\gamma\delta}^k. \quad (12)$$

Here $\tilde{\epsilon}^{\alpha\beta\gamma\delta}$ is the anti-symmetric tensor density of weight one which in any coordinate system has components ± 1 , and the proportionality means equality up to an arbitrary positive coordinate-dependent scalar factor. To fix the metric completely, it suffices to fix this factor, or to specify the associated volume form. In the case of the Plebanski theory that described GR, the associated volume form \mathcal{V} is fixed uniquely from the requirement that the metric be Einstein. The correct volume form then turns out to be:

$$3! \mathcal{V} = B^i \wedge B^i. \quad (13)$$

The content of the previous subsection can then be summarised by saying that when field equations (9)–(11) are satisfied, the metric determined by (12) with the associated volume form determined by (13) is Einstein.

The above discussion applies to complexified GR, in which case all fields in the Plebanski Lagrangian are considered to be complex-valued. Different so-called “reality conditions” are to be imposed to get different metric signatures. The easiest choice is the split signature, for which one simply takes all fields to be real. In terms of the gauge group, this corresponds to the real form $\mathrm{SL}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{C})$. The Riemannian signature is obtained by taking the real form $\mathrm{SU}(2)$. Finally, to get the metrics of Lorentzian signature, one needs to work with complex-valued fields but impose the reality conditions (here and below, an overbar denotes complex conjugation)

$$B^i \wedge \bar{B}^j = 0 \quad (14)$$

together with the condition of reality of the volume form (13). The reality conditions (14) guarantee that the complex conjugate forms \bar{B}^i are wedge product orthogonal to B^i . Given that we want to identify B^i with self-dual two-forms, and self-dual two-forms are wedge product orthogonal to anti-self-dual ones, equation (14) is the correct reality condition because it simply states that anti-self-dual two-forms are complex conjugates of self-dual ones. In other words, the reality condition (14) guarantees that the conformal metric determined by (12) is real Lorentzian.

2.5 Modified Gravity

A particular family of modified gravity theories, inspired by the Plebanski formulation, was motivated and described in [16]. In one possible parametrisation, the modification is to allow the cosmological constant in (8) to be an arbitrary $\mathrm{SO}(3)$ -invariant function of the field Ψ^{ij} :

$$S[B, A, \Psi] = \mathrm{i} \int B^i \wedge F^i - \frac{1}{2} \left(\Psi^{ij} + \frac{\Lambda(\Psi)}{3} \delta^{ij} \right) B^i \wedge B^j. \quad (15)$$

It can be shown [17] by the Hamiltonian analysis that this theory continues to propagate just two degrees of freedom, similarly to GR. At the same time, this is a modified theory of gravity, in which modification becomes important in spacetime regions where the function $\Lambda(\Psi)$ significantly deviates from a constant. A particularly simple one-parameter family of modifications is obtained by considering the function $\Lambda(\Psi)$ in the form of a quadratic polynomial in Ψ^{ij} :

$$\Lambda(\Psi) = \Lambda_0 - \frac{\alpha}{2} \mathrm{Tr}(\Psi^2), \quad (16)$$

where α is an arbitrary parameter with dimensions L^2 (i.e., inverse curvature). For this family of modified theories, one expects strong deviations from GR when the Weyl curvature Ψ becomes of the order of $1/\alpha$.

Having specified the Lagrangian, we need to say something about the metric that these modified theories might describe. As in the case of unmodified Plebanski theory, one may declare the two-forms B^i to be self-dual, which fixes the conformal class of the metric to be the Urbantke one (12). However, now there exist many different choices for the volume form. The most natural choice seems to be the one suggested by the pure-connection formulation, which is to be described below.

Lagrangian (15), with all fields taken complex-valued, describes modified complexified GR. For Riemannian and split signatures, one makes the same respective choice of reality conditions as in the unmodified Plebanski theory. Thus, in these signatures, the modified gravity theories under investigation have an honest existence. For Lorentzian signature, one might hope that choice (14) remains to be compatible with the dynamics of the theory after its modification. However, this is known to be not true in many situations. Thus, no general reality conditions are known at present that would allow for a physical interpretation of the modified theories (15). However, in some special situations, conditions (14), requiring the reality of the metric, are compatible with the dynamics. In these cases, one does obtain real metrics with Lorentzian signature as solutions to modified theories. These solutions exhibit interesting properties, and this is the reason why it looks justified to study them. Whether solutions of this kind can be imbedded into some future physical theory, to be defined by an appropriate generic choice of the reality conditions, is still an open problem.

The family of theories (15) was previously studied in [4, 5, 6, 7]. Due to a large number of independent field components present in (15), calculations with this theory are quite cumbersome. The pure-connection formulation introduced in [8] and [9] simplifies equations by eliminating most of the field components. It thus simplifies the study of these modified gravity theories considerably. The present paper is the first one where the effects of the modified gravity are analysed directly in the simpler pure-connection formulation.

2.6 An alternative parametrisation

As a step towards the pure-connection formulation, we now describe an equivalent, but slightly different parametrisation of the modified theories (15). Lagrangian (15) can be written as

$$S[B, A, M] = i \int B^i \wedge F^i - \frac{1}{2} M^{ij} B^i \wedge B^j, \quad (17)$$

where the symmetric nondegenerate matrix M^{ij} is subject to the constraint

$$\text{Tr } M = \tilde{\Lambda}(M). \quad (18)$$

Here, $\tilde{\Lambda}(M)$ is some $\text{SO}(3)$ -invariant function of the matrix M . It can be assumed without loss of generality that $\tilde{\Lambda}(M)$ only depends on the two invariants $\text{Tr}(M^2)$ and $\text{Tr}(M^3)$, because if it also depended on $\text{Tr } M$, then the constraint equation (18) written in the form

$$\text{Tr } M = \tilde{\Lambda}(\text{Tr } M, \text{Tr}(M^2), \text{Tr}(M^3)) \quad (19)$$

could be solved with respect to $\text{Tr } M$, giving birth to a new equation of the form (19), in which a new function $\tilde{\Lambda}(M)$ would be $(\text{Tr } M)$ -independent.

To make a link to the previous parametrisation, we introduce a new matrix variable Ψ via the relation

$$\Psi^{ij} = M^{ij} - \frac{1}{3} \tilde{\Lambda}(M) \delta^{ij}. \quad (20)$$

This turns constraint (18) into the relation $\text{Tr } \Psi = 0$, and establishes equivalence between theories (17), (18) and (15) with $\Lambda(\Psi(M)) = \tilde{\Lambda}(M)$. Because of this equivalence, we can identify $\Lambda(\Psi)$ and $\tilde{\Lambda}(M)$ and remove the tilde from the latter to simplify the notation.

2.7 The pure-connection formulation

The pure-connection formalism for GR, as well as for the modified theories (15), was developed in [8, 9]. Here, it will be instructive and useful to derive it from the first-order description (17), (18), which is equivalent to (15).

The pure-connection formulation of (15), or, alternatively, of (17), is obtained by integrating out all the fields apart from the connection. As the first step, one solves for the two-form fields:

$$B = M^{-1}F, \quad (21)$$

where M^{-1} is the matrix inverse of M^{ij} . Substituting this back into (17), we obtain

$$S[A, M] = \frac{i}{2} \int \text{Tr} (M^{-1}F \wedge F). \quad (22)$$

By choosing an arbitrary volume form \mathcal{V} , we define a matrix X^{ij} of scalars related to the curvature wedge products as follows:

$$F^i \wedge F^j = X^{ij} \mathcal{V}. \quad (23)$$

The action then becomes

$$S[A, M] = \frac{i}{2} \int \text{Tr} (M^{-1}X) \mathcal{V}. \quad (24)$$

Now we can integrate out the matrix M by solving its (algebraic) field equation obtained by varying this action subject to constraint (18). This leads to the pure-connection action

$$S[A] = \frac{i}{2} \int \text{Tr} (M^{-1}(X)X) \mathcal{V}. \quad (25)$$

This can alternatively be written as

$$S[A] = \frac{i}{2} \int f(X) \mathcal{V}, \quad (26)$$

which is the usual form of the pure-connection formulation as it is described, e.g., in [8]. From these expressions, we can identify the function $f(X)$ of the pure-connection formulation as

$$f(X) = \text{Tr} (M^{-1}(X)X). \quad (27)$$

Since action (25) is stationary with respect to variations of M [this gives the equation used to determine $M^{-1}(X)$], we also see that

$$\frac{\partial f}{\partial X} = M^{-1}(X). \quad (28)$$

Thus, the only remaining unsolved field equation of (17),

$$d_A (M^{-1}(X)F) = 0, \quad (29)$$

becomes the pure-connection formulation field equation

$$d_A \left(\frac{\partial f}{\partial X} F \right) = 0. \quad (30)$$

This is a second-order partial differential equation for the connection.

Note that the $\text{SO}(3)$ -invariant function $f(X)$ defined in (27) has an important property of being a homogeneous function of its argument: $f(zX) = zf(X)$ for any complex number z . This is because X enters the action (24) linearly, so that the solution $M^{-1}(X)$ in (27) is invariant with respect to a rescaling of X . In view of this property, and using definition (23), we can express the Lagrangian in (26) symbolically as

$$f(X)\mathcal{V} = f(F \wedge F). \quad (31)$$

General Relativity (with a non-zero cosmological constant Λ_0) corresponds to a particular choice of the function $f(X)$ in (26); see [9]:

$$f_{\text{GR}}(X) = \frac{1}{\Lambda_0} \left(\text{Tr} \sqrt{X} \right)^2. \quad (32)$$

Another choice of a homogeneous $\text{SO}(3)$ -invariant function $f(X)$ gives modification of General Relativity. Such modifications can be regarded as GR with arbitrary function of the curvature F added to the Lagrangian, as discussed above.

2.8 The metric from the connection

To clarify the statement that Lagrangian (32) describes GR, we should specify the metric g that solves Einstein equations when A satisfies its Euler-Lagrange equations. This can be done directly in the pure-connection language, without referring to the original Plebanski construction involving the two-form fields B^i .

As in the first-order Plebanski formalism, the metric is constructed in two steps. First, we notice that, when the field equations (21) are satisfied, the conformal class of metrics in which B^i are self-dual is the same as the one in which F^i are self-dual, since both sets of two-forms span the same three-dimensional subspace in the space of all two-forms. Thus, instead of using (12), we can obtain the conformal class of the metric directly from the curvature of the connection:

$$g_{\alpha\beta} \sqrt{\det g} \sim \tilde{\epsilon}^{\mu\nu\rho\sigma} \epsilon_{ijk} F_{\mu\nu}^i F_{\rho\alpha}^j F_{\sigma\beta}^k. \quad (33)$$

For complex-valued connection components A^i , the conformal class obtained is, in general, complex (i.e., does not contain any real-valued metric). The reality conditions that ensure that the conformal class is that of a real Lorentzian metric can be stated as

$$F^i \wedge \bar{F}^j = 0. \quad (34)$$

All this parallels the above discussion; we have only replaced the two-forms B^i by F^i .

In the second step, one specifies a particular metric in the conformal class already defined. To do so, it suffices to fix the associated volume form. The volume form \mathcal{V}_A that gives the metric satisfying the Einstein equations in the case of theory (32) is determined by

$$-2i\Lambda_0^2\mathcal{V}_A = \Lambda_0 f_{\text{GR}}(F \wedge F). \quad (35)$$

The factor of the imaginary unit is needed in this relation for the same reason as in (26). Thus, in the case of pure-connection description of GR, the action of the theory (26) is just the total volume of the space, which is very appealing.

For a modified gravity theory with a general function $f(F \wedge F)$ as a Lagrangian, we do not yet know what the correct “physical” metric would be. However, by analogy with GR, we could assume that the action is again the total volume calculated with respect to this metric. This would imply that the metric volume form is a multiple of $f(F \wedge F)$. It is this metric that we are going to study below.

2.9 Mixed parametrisation

It is not hard to pass from the Plebanski formulation of GR with $\Lambda \neq 0$ to the pure-connection formulation of GR and to obtain the Lagrangian function (32). It turns out to be surprisingly hard, however, to characterise modifications even of a simple type (16) in the pure-connection language. It is also clear from (32) that the pure-connection description involves the operation of taking the square root of a matrix, which requires specifying its branch. Another issue with the pure-connection formulation is that it only makes sense for $\Lambda \neq 0$.

To overcome these difficulties, in this subsection we introduce a new, “mixed” parametrisation that lies half-way between the Plebanski first-order description (17) and the pure-connection description (26) and that combines the advantages of both. The idea is to avoid solving the field equations for M , and work with both M and A as independent variables.

Let us see how this can work in practice. As a first step, we replace the constrained variational problem for M in (24) by an unconstrained one, with an additional Lagrange multiplier μ imposing the constraint:

$$S[A, M, \mu] = \frac{i}{2} \int [\text{Tr}(M^{-1}X) + \mu(\text{Tr}(M) - \Lambda(M))] \mathcal{V}. \quad (36)$$

Variation with respect to M then gives the equation

$$M^{-1}X = \mu \left(M - \frac{\partial \Lambda(M)}{\partial M} M \right). \quad (37)$$

To get to the pure-connection formulation, we are supposed to solve this, and the constraint equation (18), for M and μ in terms of X . However, this is very difficult to do in general. Only in the case of GR, where Λ does not depend on M , one can do this and immediately obtain (32). For this reason, let us instead solve equation (37) with respect to X . Taking

the trace of this equation, and using the constraint on M , as well as definition (27), we can eliminate the Lagrange multiplier μ and get

$$X = f(X) \frac{M^2 - M \frac{\partial \Lambda(M)}{\partial M} M}{\Lambda(M) - \text{Tr} \left[\frac{\partial \Lambda(M)}{\partial M} M \right]}. \quad (38)$$

Thus, the matrix X is determined by M up to scale. Then, if we interpret the field equation

$$d_A(M^{-1}F) = 0 \quad (39)$$

as a partial differential equation for M and solve it, we will find X from (38). This is the strategy we will follow below to solve the field equations in the case of homogeneous anisotropic cosmology.

Let us also give the form of relation (38) in the case of parametrisation by using variable (20) and function $\Lambda(\Psi)$ in (15). We have

$$X = f(X) \frac{(\Psi + \frac{1}{3}\Lambda(\Psi)\text{Id})^2 (\text{Id} - \frac{\partial \Lambda}{\partial \Psi})}{\Lambda(\Psi) - \text{Tr} \left(\frac{\partial \Lambda}{\partial \Psi} \Psi \right)}, \quad (40)$$

where Id denotes the identity matrix.

The case $\Lambda(\Psi) \equiv 0$, which corresponds to GR without the cosmological constant, needs to be considered separately. This theory does not have pure-connection formulation, but one can still solve the field equations in the mixed parametrisation proposed here. In this case, equation (37) gives

$$\Psi^{-1}X = \frac{1}{3}\Psi \text{Tr}(\Psi^{-2}X). \quad (41)$$

In particular, this equation implies that the function $f(X) = \text{Tr}(\Psi^{-1}(X)X) \equiv 0$ in the case of GR without the cosmological constant. We analyse the Bianchi I cosmology for this case in the Appendix.

In either of the two cases, one needs to solve the coupled system of equations (38), (39) or (41), (39) (with $M = \Psi$) and find the components of the connection. So, this is still a connection formulation with the connection playing the role of the main field. However, there are now additional auxiliary fields Ψ . One of the benefits of this mixed formulation is that, even in the case of GR, there is no need to take square roots. And, as we shall see below, the coupled system of equations can be solved, at least in special situations.

2.10 Gravitational Instantons

Before we proceed to the analysis of the homogeneous anisotropic field configurations as described by our modified gravity theories, we make one further remark about the gravitational instantons in our formalism. This subsection can be skipped on the first reading.

Let us rewrite the mixed parametrisation (36) of our theories as

$$S[A, M, \mu] = \frac{i}{2} \int [\text{Tr} (M^{-1} X) + \mu g(M)] \mathcal{V}, \quad (42)$$

where we have introduced a scalar-valued function of matrix M :

$$g(M) := \text{Tr} M - \Lambda(M). \quad (43)$$

The role of the Lagrange multiplier μ is to impose a single constraint on the matrix M^{ij} . This constraint can be viewed as defining a codimension-one surface $g(M) = 0$ in the space of matrices M . By the Hamiltonian analysis of this class of theories, this constraint can be shown to be, in fact, the Hamiltonian constraint in disguise. As we have seen above, the function

$$g_{\text{GR}}(M) = \text{Tr} M - \Lambda_0 \quad (44)$$

with constant Λ_0 gives the description of GR with the cosmological constant Λ_0 .

We now note that, M being an auxiliary field, any field redefinition of M in (42) should give an equivalent description of the theory. Thus, in particular, the Lagrangian

$$S[A, \tilde{M}, \mu] = \frac{i}{2} \int [\text{Tr} (\tilde{M} X) + \mu \tilde{g}(\tilde{M})] \mathcal{V}, \quad (45)$$

where $\tilde{M} = M^{-1}$, and $\tilde{g}(\tilde{M}) = g(\tilde{M}^{-1})$ should provide an equivalent description.

It is then interesting to note that, by taking

$$\tilde{g}_{\text{self-dual}}(\tilde{M}) = \text{Tr} \tilde{M} - \Lambda_0 \quad (46)$$

with constant Λ_0 , one obtains a Lagrangian describing self-dual gravity. Indeed, varying (45) with respect to \tilde{M} , we obtain the equations

$$X^{ij} \sim \delta^{ij}, \quad (47)$$

which are known to provide the correct description of the gravitational instantons in the pure-connection formalism. In terms of the original action (42), this means choosing the function $g(M)$ in the form

$$g_{\text{self-dual}}(M) = \text{Tr} (M^{-1}) - \Lambda_0. \quad (48)$$

Thus, our mixed parametrisation is flexible enough to incorporate GR, modified gravity theories, and self-dual gravity. One just constraints the auxiliary matrix field onto different codimension-one surfaces. It is interesting that GR (44) and the instanton sector (48) go one into another by the replacement $M \rightarrow M^{-1}$ in the constraint equation.

3 Bianchi I connections

We now start exploring how the familiar general-relativistic Kasner solution arises in the above pure-connection formulation, and what the modifications of gravity entail.

The following ansatz for the Bianchi I connection is motivated in the Appendix:

$$A^i = i h_i(\tau) dx^i. \quad (49)$$

Note that there is no summation over i on the right-hand-side of this equation. Here, $h_i(\tau)$ are three functions of an arbitrary time coordinate τ , while x^i are the Cartesian coordinates on the spatial slices (surfaces of homogeneity). The corresponding curvature two-form is

$$F^i = dA^i + \frac{1}{2} \epsilon^{ijk} A_j \wedge A_k = i \dot{h}_i d\tau \wedge dx^i - \frac{1}{2} \epsilon^{ijk} h_j h_k dx^j \wedge dx^k, \quad (50)$$

where an overdot denotes derivative with respect to τ . Calculating the wedge product, we obtain

$$F^i \wedge F^j = -2i \delta^{ij} X_i h \mathcal{V}_c, \quad (51)$$

where $\mathcal{V}_c = d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3$ is the coordinate volume form, $h = h_1 h_2 h_3$, and

$$X_i = \frac{\dot{h}_i}{h_i}. \quad (52)$$

If we select the volume form \mathcal{V} in (23) to be

$$\mathcal{V} = -2i h \mathcal{V}_c, \quad (53)$$

then $X^{ij} = \text{diag}(X_1, X_2, X_3)$.

3.1 Evolution equations in the pure-connection parametrisation

The reason for the above choice of the volume form defining the matrix X^{ij} is that the pure-connection formulation equation (30) reduces to the system

$$\left(\frac{\partial f}{\partial X_i} \right)' = f(X) - \frac{\partial f}{\partial X_i} \sum_j X_j, \quad (54)$$

which is a system of first-order differential equations for X_i . A derivation is given in the Appendix (in more generality), but, for the present situation with diagonal matrix X , it is quite easy to obtain these equations directly. Specialising to the case of the function $f(X)$ given by (32), it is not hard to obtain the familiar GR solution; see Appendix.

An alternative form of equations (54) is obtained by multiplying these equations by $h = h_1 h_2 h_3$ and using definition (52). Equations (54) then reduce to

$$\left(\frac{\partial f}{\partial X_i} h \right)' = f(X) h. \quad (55)$$

This form will be very convenient for analysing the case of arbitrary $f(X)$.

3.2 Evolution equations in the parametrisation with auxiliary fields

Let us also give the form of the evolution equations arising in the mixed parametrisation, which involves both the connection and the matrix M . The equation in question is then (39). Given that the matrix M is the inverse of the matrix of first derivatives of the function $f(X)$ [see (28)], it is diagonal whenever X is diagonal: $M^{ij} = \text{diag}(M_1, M_2, M_3)$. Using relation (27), one can write equations (39) in a form similar to (55):

$$(M_i^{-1}h)' = \text{Tr} (M^{-1}X) h. \quad (56)$$

This form will be most convenient for the analysis of modifications in the parametrisation by a function $\Lambda(\Psi)$.

3.3 The metric

Before we begin our analysis of the evolution equations given above, it is useful to compute the metric determined by the connection. To this end, one can follow the previously described procedure by first computing the Urbantke metric (33) and then conformally rescaling it so that the volume form becomes a multiple of $f(F \wedge F)$. A simpler method is to directly look for a metric that makes the curvature forms (50) self-dual.

We are looking for the metric in the Bianchi I form

$$ds^2 = -N^2(\tau)d\tau^2 + \sum_i a_i^2(\tau) (dx^i)^2. \quad (57)$$

Calculating the dual $*F^i$ of the curvature two-forms (50) with respect to this metric, we have

$$*F^i = -i\dot{h}_i \frac{Na_1a_2a_3}{2N^2a_i^2} \epsilon_{ijk} dx^j \wedge dx^k - \frac{1}{2} \epsilon^{ijk} h_j h_k \frac{Na_1a_2a_3}{a_j^2 a_k^2} \epsilon_{jkl} d\tau \wedge dx^l. \quad (58)$$

The requirement $*F^i = \pm iF^i$ gives the condition

$$\pm \dot{h}_1 = \frac{Na_1}{a_2a_3} h_2 h_3, \quad \text{etc}, \quad (59)$$

from which we get

$$\frac{a_1^2}{N^2} = \frac{h_1^2}{X_2 X_3}, \quad \text{etc}. \quad (60)$$

Another equation for determining the metric is obtained by fixing the metric volume form

$$\mathcal{V}_m = Na_1a_2a_3 d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3 = Na_1a_2a_3 \mathcal{V}_c. \quad (61)$$

By our prescription, this should be equal to a multiple of $f(F \wedge F)$:

$$-2i\Lambda_0 \mathcal{V}_m = f(F \wedge F), \quad (62)$$

where Λ_0 is some parameter of dimension $1/L^2$, later to be identified with the cosmological constant. Using (51), we get

$$\Lambda_0 N a_1 a_2 a_3 = f(X) h. \quad (63)$$

Combining this equation with (60), we have

$$N^2 = \left(\frac{f^2(X) \prod_i X_i^2}{\Lambda_0^2} \right)^{1/4}, \quad a_1^2 = N^2 \frac{h_1^2}{X_2 X_3}, \quad \text{etc}, \quad (64)$$

where the first relation is written in the form indicating that there are, in general, four possible branches, two of them imaginary. If we require that the metric be real, and that the signature of the τ coordinate be negative, then the final expression for the metric is

$$ds^2 = \sqrt{\left| \frac{f(X) \prod_i X_i}{\Lambda_0} \right|} \left[-d\tau^2 + \prod_j X_j^{-1} \sum_k h_k^2 X_k (dx^k)^2 \right]. \quad (65)$$

Note that this metric is time-reparametrisation invariant, as it should be.

4 Solution in the general case

One of the miracles of the pure-connection formulation of gravity under consideration is that it allows one to write the *general* solution to the problem at hand for an arbitrary theory, i.e., for an arbitrary choice of the function $f(X)$. This becomes possible by using a clever choice of the time variable.

We begin with solution for the case of general $f(X)$ and then specialise to GR and to a particular modified gravity theory. Solution of GR in which one works in the physical time from the beginning is also possible. For completeness, it is presented in the Appendix.

In the previous section, we arrived at evolution equations in the form (55). By using time-reparametrisation freedom, it is always possible to choose the time variable τ in such a way that

$$fh = \text{const}. \quad (66)$$

The geometric significance of this choice is that this is the time coordinate in which the metric volume form is proportional to the coordinate volume form, i.e., $\sqrt{\det g} = N a_1 a_2 a_3 = \text{const}$. This is clear from (63).

With this choice, equation (55) can be integrated to give an implicit solution for $X(\tau)$:

$$\frac{\partial f(X)}{\partial X_i} = f(X) (\tau - \tau_i), \quad (67)$$

where τ_i are arbitrary integration constants. The homogeneity of the function $f(X)$ implies another relation

$$\sum_i X_i (\tau - \tau_i) = 1. \quad (68)$$

Equations (67) and (52) give a complete solution to the problem for an arbitrary theory from our class. We now give some general analysis of the solution obtained, and then consider some specific functions $f(X)$.

4.1 De Sitter solution

Consider $\tau \rightarrow \infty$, and assume that $f(X)\tau$ remains constant as $\tau \rightarrow \infty$. Then equation (67) implies that all derivatives $\partial f(X)/\partial X_i$ become mutually equal. The symmetry of the function $f(X)$, in turn, implies that all X_i become equal to each other in this limit. Relation (68) then gives the solution

$$X_i \approx \frac{1}{3\tau} \quad \text{as } \tau \rightarrow \infty. \quad (69)$$

The homogeneity of $f(X)$ then justifies the assumption $f(X)\tau \rightarrow \text{const}$ that we made in deriving this solution.

The corresponding metric describes the De Sitter spacetime. Indeed, we have $f(X) = f_0/\tau$, where f_0 is a constant. Then, by rescaling the spatial coordinates, we can always choose the solution in the form $h_i = \tau^{1/3}$. Then metric (65) becomes

$$ds^2 = \sqrt{\frac{3f_0}{\Lambda_0}} \left(-\frac{d\tau^2}{9\tau^2} + \tau^{2/3} dr^2 \right) = \sqrt{\frac{3f_0}{\Lambda_0}} (-dt^2 + e^{2t} dr^2), \quad (70)$$

where $\tau = e^{3t}$ is the time coordinate change, and $dr^2 = \sum_i (dx^i)^2$. This is nothing but the De Sitter metric, which is thus the solution of any modified theory.

4.2 Integration constants

Without loss of generality, one can shift the time variable so that

$$\sum_i \tau_i = 0. \quad (71)$$

Second, apart from the trivial case $\tau_i = 0$ for all i , which gives the De Sitter solution, by the remaining freedom of time rescaling, which does not violate (66), we can achieve the condition

$$\sum_i \tau_i^2 = 2. \quad (72)$$

This normalization is convenient because squaring (71) we can rewrite (72) as

$$\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = -1. \quad (73)$$

Without loss of generality, we can arrange the integration constants so that

$$\tau_3 \leq \tau_2 \leq \tau_1. \quad (74)$$

Because of condition (71), we have $\tau_3 < 0 < \tau_1$. When $\tau_2 = \tau_1$, we have $\tau_2 = \tau_1 = 1/\sqrt{3}$ and $\tau_3 = -2/\sqrt{3}$. This is the largest absolute value that τ_3 can reach. In the opposite extreme $\tau_2 = \tau_3$ we have $\tau_2 = \tau_3 = -1/\sqrt{3}$ and $\tau_1 = 2/\sqrt{3}$, which is the largest value τ_1 can reach. All in all, we have

$$\tau_c \leq \tau_1 \leq 2\tau_c, \quad -\tau_c \leq \tau_2 \leq \tau_c, \quad -2\tau_c \leq \tau_3 \leq -\tau_c, \quad (75)$$

where $\tau_c = 1/\sqrt{3}$.

4.3 Solution in parametrisation with auxiliary matrix M

From now on, we will work in the mixed parametrisation with auxiliary matrix M . The evolution equation in this case takes the form (56). Choosing the same time variable as above, namely, the one that satisfies (66), we immediately obtain the solution

$$M^{-1} = f(X)T, \quad (76)$$

where $T^{ij} = \delta^{ij}(\tau - \tau_i)$ is another diagonal matrix. This, in fact, is the same solution as (67); we simply refrained from solving the relation between M and X . From this, we have

$$\text{Tr } M = \Lambda(M) = \frac{\text{Tr}(T^{-1})}{f(X)}, \quad M = \Lambda(M) \frac{T^{-1}}{\text{Tr}(T^{-1})}. \quad (77)$$

Equation (38) then becomes

$$X = \frac{T^{-2} \left(\text{Id} - \frac{\partial \Lambda(M)}{\partial M} \right)}{\text{Tr} \left[T^{-1} \left(\text{Id} - \frac{\partial \Lambda(M)}{\partial M} \right) \right]}. \quad (78)$$

The second equation in (77) enables one to find M as a function of time. Substituting it into (78), one finds the solution for X .

4.4 Solution in parametrisation with auxiliary matrix Ψ

By expressing the partial derivatives in (78) in terms of Ψ introduced in (20), one easily obtains a solution in the form

$$X = \frac{T^{-2} \left(\text{Id} - \frac{\partial \Lambda(\Psi)}{\partial \Psi} \right)}{\text{Tr} \left[T^{-1} \left(\text{Id} - \frac{\partial \Lambda(\Psi)}{\partial \Psi} \right) \right]}. \quad (79)$$

In the Ψ parametrisation, the second equation in (77) is written as

$$\Psi = \Lambda(\Psi) \left[\frac{T^{-1}}{\text{Tr}(T^{-1})} - \frac{1}{3} \text{Id} \right]. \quad (80)$$

It is to be solved with respect to Ψ , with the result to be substituted into (79).

5 The case of GR

Here, we obtain the solution of GR in this time variable. In the case $\Lambda(\Psi) \equiv \Lambda_0 = \text{const}$, one gets

$$X_i^{\text{GR}} = \frac{1}{s_1(\tau - \tau_i)^2}, \quad \text{or} \quad X_1^{\text{GR}} = \frac{(\tau - \tau_2)(\tau - \tau_3)}{(3\tau^2 - 1)(\tau - \tau_1)}, \quad \text{etc}, \quad (81)$$

where

$$s_1 \equiv \text{Tr}(T^{-1}) \equiv \sum_i \frac{1}{\tau - \tau_i} = \frac{3\tau^2 - 1}{\prod_i (\tau - \tau_i)}, \quad (82)$$

and we have used both (71) and (72). The quantities X_i^{GR} have simple poles at $\tau = \tau_i$, and all blow up as $\tau \rightarrow \pm 1/\sqrt{3}$, which corresponds to the Kasner singularity. This behaviour is illustrated in Fig. 1.

Let us also write the corresponding metric components; see (64). According to (77), we have

$$f(X) = \frac{s_1}{\Lambda_0}. \quad (83)$$

Thus,

$$\frac{f(X)X_1X_2X_3}{\Lambda_0} = \frac{1}{\Lambda_0^2(3\tau^2 - 1)^2}, \quad \frac{f(X)X_1}{\Lambda_0X_2X_3} = \frac{(3\tau^2 - 1)^2}{\Lambda_0^2(\tau - \tau_1)^4}, \quad (84)$$

and similarly for the other components. All expressions are manifestly positive, so taking the square root, we have

$$N^2 = \frac{1}{\Lambda_0(3\tau^2 - 1)}, \quad a_i^2 = h_i^2 \frac{3\tau^2 - 1}{\Lambda_0(\tau - \tau_1)^2}. \quad (85)$$

In the time interval $\tau \in (-\tau_c, \tau_c)$, $\tau_c = 1/\sqrt{3}$, instead of taking the modulus of expressions to get non-negative metric components, we reverse the sign of the cosmological constant Λ_0 . This is the correct interpretation, as this time interval corresponds to a solution of GR with negative cosmological constant.

We now study this solution in more detail, and, in particular, integrate the equations for h_i near the singularity.

5.1 Behaviour near the poles

When all three integration constants are different, the function $s_1(\tau)$ has three simple poles at $\tau = \tau_i$, and two simple zeros at $\tau = \pm\tau_c$. Let us analyse the behavior near the poles.

Consider, for example, the limit $\tau \rightarrow \tau_1$. In this case, we have $s_1 \sim 1/(\tau - \tau_1) \rightarrow \infty$. Solution (81) behaves as

$$X_1 \approx \frac{1}{\tau - \tau_1}, \quad X_2 \sim X_3 \sim \tau - \tau_1. \quad (86)$$

This is an integrable behavior, with $h_1 \rightarrow 0$ and h_2 and h_3 finite as $\tau \rightarrow \tau_1$. We thus see that all X_i change sign at $\tau = \tau_1$.

Let us determine the behaviour of the components (64) of the canonical metric (65) at this point. Integrating the first equation in (86), we obtain

$$h_1 \sim \tau - \tau_1, \quad (87)$$

while h_2 and h_3 tend to constants. So, the significance of the point $\tau = \tau_1$ is in the fact that one of the connection components h_1 passes through zero there.

Now, using this behaviour we see that the metric lapse function as well as the scale factors (64) are finite and regular as $\tau \rightarrow \tau_1$. So, the $\tau = \tau_1$ is just a special point in the solution.

5.2 Behaviour near the singularity

At the singularity $\tau = \tau_c = 1/\sqrt{3}$, the function s_1 has a simple zero. Thus, we have

$$X_1 \approx -\frac{(\tau_c - \tau_2)(\tau_c - \tau_3)}{2\sqrt{3}(\tau_1 - \tau_c)(\tau - \tau_c)}, \quad \text{etc}, \quad f(X) \sim \tau - \tau_c. \quad (88)$$

Integrating (52), we get

$$h_1 \sim (\tau - \tau_c)^{-\frac{(\tau_c - \tau_2)(\tau_c - \tau_3)}{2\sqrt{3}(\tau_1 - \tau_c)}}, \quad \text{etc}. \quad (89)$$

We thus see that the lapse function (85) diverges, while the scale factors behave as

$$a_i^2 \sim (\tau - \tau_c)^{p_i}, \quad (90)$$

where

$$p_1 = 1 - \frac{(\tau_c - \tau_2)(\tau_c - \tau_3)}{\sqrt{3}(\tau_1 - \tau_c)}, \quad \text{etc}. \quad (91)$$

These exponents satisfy

$$p_1 + p_2 + p_3 = 1, \quad p_1 p_2 + p_2 p_3 + p_3 p_1 = 0. \quad (92)$$

From (85) we see that the physical time near the singularity is $t \sim \sqrt{\tau - \tau_c}$, and thus the behaviour (90) is the usual Kasner one,

$$a_i^2 \sim t^{2p_i}, \quad (93)$$

with the correct exponents (92).

Note that the components of the gauge field (89) all diverge at the singularity, so this is a true singularity not only of the canonical metric (65) but also of the fundamental gauge field.

5.3 Summary of the GR solution

We summarise the facts established above. As $\tau \rightarrow \infty$, we approach the De Sitter solution $X_i = 1/3\tau$. As time decreases, at $\tau = \tau_1$ we encounter a special point where X_1 has a simple pole, while X_2 and X_3 vanish. Below this point, all X_i change sign, as does $f(X)$. The component of the connection h_1 vanishes at this point, while the components h_2 and h_3 remain finite. All components of the canonical metric (65) remain finite at this point.

As $\tau \rightarrow \tau_c$, we approach the Kasner singularity, with the functions X_i all negative near the singularity, and all having a simple pole there. The function N^2 also has a simple pole at this point. The scale factors a_i^2 exhibit the familiar Kasner behaviour (90).

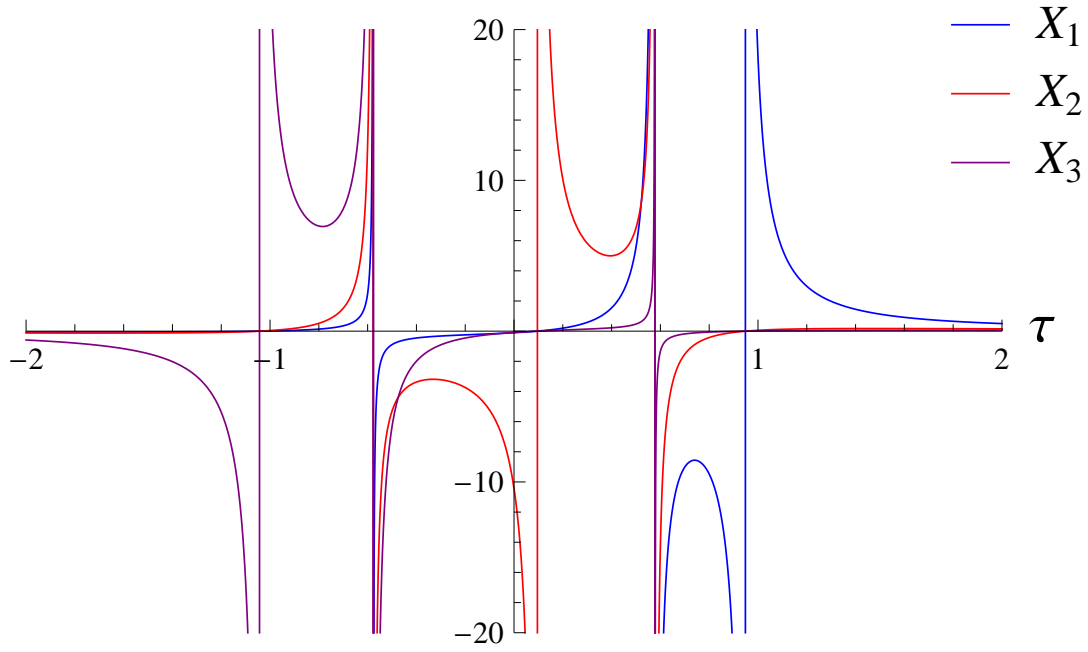


Figure 1: Behaviour of the variables X_i given by (81) in General Relativity. Each variable X_i has a pole at respective $\tau = \tau_i$. These are just special points of the solution, with all connection and metric components remaining finite. Furthermore, all variables X_i have common poles at $\tau = \pm\tau_c$. These are the singular points for both the connection and the metric.

We also note that the region $\tau > \tau_1$ is where the solution is guided by the cosmological constant. Near the Kasner singularity, as $\tau \rightarrow \tau_c$, the Weyl curvature becomes so strong that the cosmological constant does not play any role and can be neglected.

Since the gauge field diverges at the singularity, the domain $-\tau_c < \tau < \tau_c$ in (81) can only be treated as another singular solution. Let us first consider the region $\tau_2 < \tau < \tau_c$. In order that the metric with components (85) be of the usual signature there, we need to change the sign of the cosmological constant Λ_0 . Thus, the solution in this time interval is described by GR with negative cosmological constant. The behaviour near τ_c is again Kasner. The point τ_2 again is a special point of the solution, in which X_2 has a simple pole, while X_1 and X_3 have simple zeros. Thus, h_2 passes through zero at this point, with h_1 and h_3 remaining finite and nonzero. The metric components are all finite and non-zero. As $\tau \rightarrow -\tau_c$, we encounter another Kasner singularity. Thus, the $\Lambda_0 < 0$ part of the solution interpolates between two Kasner singularities. There is no asymptotic anti-De Sitter regime in this case.

For $\tau < -\tau_c$, we have another copy of asymptotically De Sitter solution. We note that all X_i are positive near the singularity in this case, as is $f(X)$. There is a Kasner singularity as $\tau \rightarrow -\tau_c$, and a special point at $\tau = \tau_3$ with h_3 vanishing and all X_i and $f(X)$ changing sign. As $\tau \rightarrow -\infty$, we approach another De Sitter region. Since the time change $\tau \rightarrow -\tau$ makes the region $\tau < -\tau_c$ mathematically equivalent to the asymptotically De Sitter region $\tau > \tau_c$ discussed above, it is clear that the Kasner exponents near the singularity in the region $\tau < -\tau_c$ are obtained from (91) by the replacement $\tau_i \rightarrow -\tau_i$.

6 One-parameter family of modifications

6.1 Modified $\Lambda_0 > 0$ case

Consider the one-parameter family (16) with $\Lambda_0 > 0$ and small modification $\alpha\Lambda_0 \ll 1$. In this case, one expects that modification only becomes significant in the region where the Weyl curvature is large, i.e., near the singularity. This is correct, and, as we will see below, the singularity at $\tau = \pm\tau_c$ gets resolved in the modified theory.

To find solution for X , we need to solve equation (80) for $\Psi(\tau)$ and substitute the result into (79). Since, by virtue of (80), $\Psi(\tau)$ is determined by $\Lambda(\tau)$, it will be sufficient to find $\Lambda(\tau)$. To find X from (79) in theory (16), we calculate

$$\text{Id} - \frac{\partial \Lambda}{\partial \Psi} = \text{Id} + \alpha \Psi = \text{Id} + \alpha \Lambda \left(\frac{T^{-1}}{s_1} - \frac{1}{3} \text{Id} \right), \quad (94)$$

$$\text{Tr} \left[T^{-1} \left(\text{Id} - \frac{\partial \Lambda(\Psi)}{\partial \Psi} \right) \right] = s_1 \left[1 + \alpha \Lambda \left(\frac{s_2}{s_1^2} - \frac{1}{3} \right) \right], \quad (95)$$

where, in addition to (82), we have introduced the function

$$s_2 = \text{Tr} (T^{-2}) = \sum_i \frac{1}{(\tau - \tau_i)^2} = \frac{3\tau^4 + 6\tau_1\tau_2\tau_3\tau + 1}{\prod_i (\tau - \tau_i)^2}. \quad (96)$$

Eventually, we have

$$X = \frac{(1 - \frac{1}{3}\alpha\Lambda)s_1 T^{-2} + \alpha\Lambda T^{-3}}{s_1^2(1 + \alpha x\Lambda)}, \quad (97)$$

where

$$x(\tau) \equiv \left(\frac{s_2}{s_1^2} - \frac{1}{3} \right) = 2 \frac{3\tau^2 + 9\tau_1\tau_2\tau_3\tau + 1}{3(3\tau^2 - 1)^2}. \quad (98)$$

It remains to find $\Lambda(\tau)$. Using (80), we have

$$\Psi^2 = \Lambda^2 \left(\frac{T^{-2}}{s_1^2} - \frac{2T^{-1}}{3s_1} + \frac{1}{9}\text{Id} \right), \quad \text{Tr}(\Psi^2) = \Lambda^2 \left(\frac{s_2}{s_1^2} - \frac{1}{3} \right), \quad (99)$$

and equation (16) then produces the quadratic equation for $\Lambda(\tau)$:

$$-\frac{\alpha}{2}x(\tau)\Lambda^2 = \Lambda - \Lambda_0. \quad (100)$$

Consider first the region $|\tau| > \tau_c$. (The region $|\tau| < \tau_c$ is analysed in Section 6.4.) Note that the function $x(\tau)$, defined in (98), is nonnegative and approaches infinity as $|\tau| \rightarrow \tau_c$. Therefore, in order that the quadratic equation (100) always have a real solution for Λ , it is necessary to demand that $\alpha > 0$. As $|\tau| \rightarrow \infty$, the function $x(\tau)$ tends to zero. In order that the solution for Λ tend to the general-relativistic value Λ_0 in this limit, one should take the positive root of the quadratic equation (100):

$$\Lambda = \Lambda_+ = \frac{1}{\alpha x} \left(\sqrt{1 + 2\alpha x\Lambda_0} - 1 \right). \quad (101)$$

Since x and Λ are both non-negative in the region $|\tau| > \tau_c$, the only place where we possibly can encounter singularity in solution (97) is when s_1 turns to infinity or zero. The first option occurs as $\tau \rightarrow \tau_1$, at which point the diagonal matrix T^{-1} also becomes singular. The second option occurs as $|\tau| \rightarrow \tau_c = 1/\sqrt{3}$ (this is a singular point in GR). Let us consider these two possibilities separately.

6.2 Behaviour near the pole

As $\tau \rightarrow \tau_1$, we have $s_1 \approx 1/(\tau - \tau_1) \rightarrow \infty$, $x \rightarrow 2/3$, and, assuming $\alpha\Lambda_0 \ll 1$ (small modification), $\Lambda \rightarrow \Lambda(\tau_1) \approx \Lambda_0$. Solution (97) behaves as in GR and the modification is negligible. This is a special point of the solution, with all X_i changing sign there (and one of them, X_1 , having a simple pole). All metric components are finite and non-zero there.

6.3 Behaviour near the would-be singularity

Consider now the critical point. As $\tau \rightarrow \tau_c$ from above, we have $s_1 \rightarrow 0$ and $x \rightarrow \infty$, so that $\Lambda \approx \sqrt{2\Lambda_0/\alpha x} \rightarrow 0$. We also have $x \approx s_2/s_1^2$ with s_2 being finite at this point. Therefore, the denominator in (97) behaves as

$$s_1^2(1 + \sqrt{2\alpha\Lambda_0 x}) \approx -s_1\sqrt{2\alpha\Lambda_0 s_2}, \quad (102)$$

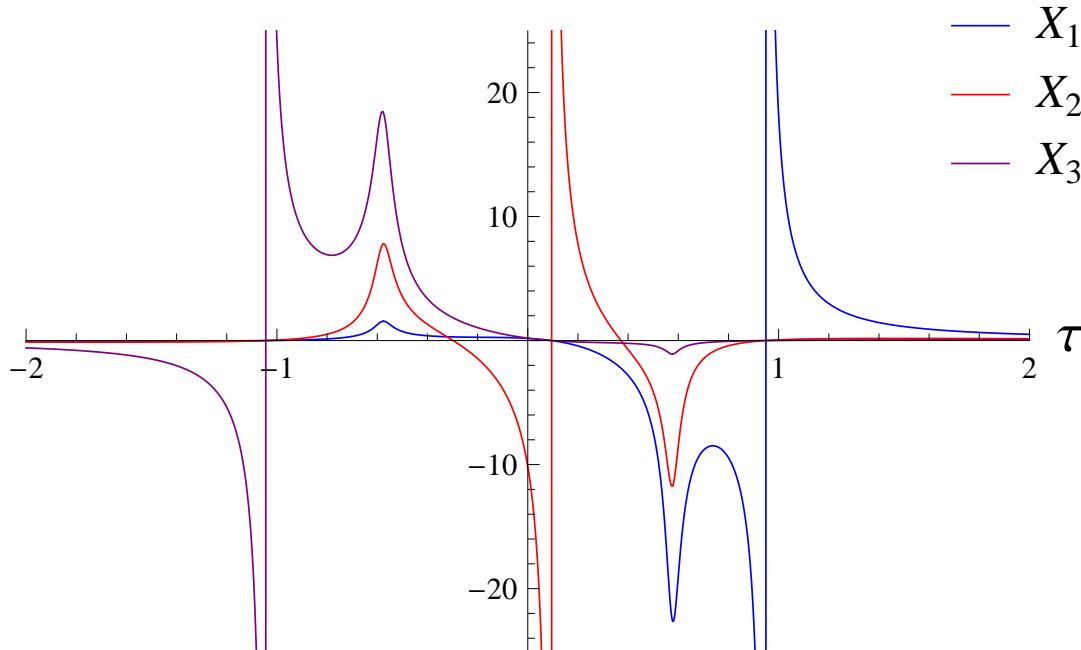


Figure 2: Behaviour of the variables X_i given by (97) in the modified theory (16) with $\alpha\Lambda_0 = 5 \times 10^{-3}$. As in General Relativity (see Fig. 1), each variable X_i has a pole at respective $\tau = \tau_i$. However, the common poles at $\tau = \pm\tau_c$ disappear, and the variables X_i become regular at these points.

where we have taken into account that $s_1 < 0$ as we approach τ_c from above. The numerator in (97) behaves as $s_1 T^{-2}$, with small correction of order $\sqrt{\alpha\Lambda_0}$ that can be neglected. Solution (97) then behaves as

$$X_i \approx -\frac{1}{\sqrt{2\alpha\Lambda_0 s_2}(\tau - \tau_i)^2} \sim -\frac{1}{\sqrt{\alpha\Lambda_0}}, \quad (103)$$

which is large by absolute value (under the assumption $\alpha\Lambda_0 \ll 1$) but finite.

6.4 Behaviour “inside” the would-be singularity

In the general-relativistic solution, all X_i are negative and blow up near the Kasner singularity $\tau \rightarrow \tau_c$. In the previous subsection we have seen that modification resolves this singularity, making all X_i large negative but finite. Our solution thus smoothly continues to the region $\tau < \tau_c$. Since X_i are finite and nonzero at this point, $f(X)$ is also continuous. The function s_1 changes sign from negative to positive as one crosses τ_c from above. In view of the relation $f(X) = s_1/\Lambda$ [see (77)], this means that the sign of Λ should become negative below the point τ_c . To ensure this, we should take the negative root of (100) in

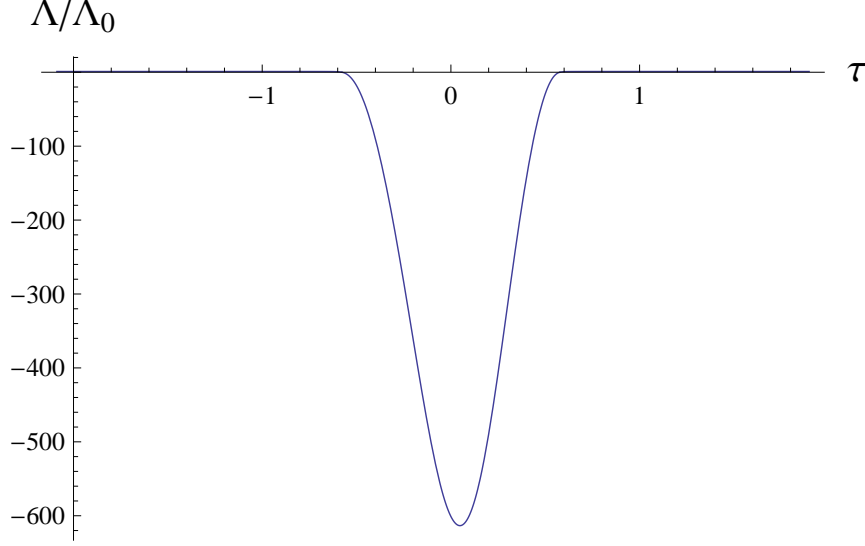


Figure 3: Behaviour of $\Lambda(\tau)/\Lambda_0$ in the modified theory (16) with $\alpha\Lambda_0 = 5 \times 10^{-3}$.

the region $|\tau| < \tau_c$:

$$\Lambda = \Lambda_- = -\frac{1}{\alpha x} \left(\sqrt{1 + 2\alpha x \Lambda_0} + 1 \right), \quad (104)$$

with x defined in (98). Note that this branch of the solution does not have the GR limit as $\alpha \rightarrow 0$. Also note that the denominator in (97) is always negative on this branch. One can easily check that solution (97) is continuously differentiable at the point $\tau = \tau_c$.

As time decreases from τ_c , the next special point that we encounter is $\tau = \tau_2$. Around this point, $s_1 \approx 1/(\tau - \tau_2)$, while $x \rightarrow 2/3$. Now we have $\Lambda \rightarrow \Lambda_-(\tau_2) \approx -3/\alpha$, which is large by absolute value compared to Λ_0 . In the neighbourhood of $\tau = \tau_2$, solution (97) is then approximated as

$$X = \frac{3T^{-3} - 2s_1T^{-2}}{s_1^2}. \quad (105)$$

We observe that X_1 and X_3 cross zero at $\tau = \tau_2$, while $X_2 \approx 1/(\tau - \tau_2)$ has a simple pole, approaching positive infinity as $\tau \rightarrow \tau_2$ from above. Since all X_i were negative at $\tau = \tau_c$, this means that X_2 crosses zero at some $\tau_+ \in (\tau_2, \tau_c)$. It then crosses zero once again at some $\tau_- \in (-\tau_c, \tau_2)$. This behaviour is demonstrated in Fig. 2. In the interval (τ_-, τ_+) , metric (65) changes signature from $(-, +, +, +)$ to $(-, -, +, -)$, the spatial coordinate x^2 thus taking the role of time. Thus, although we do not encounter singularity in the fundamental gauge field (all h_i are everywhere smooth), there is a singularity in metric (65) at the points $\tau = \tau_{\pm}$, where it also changes signature.

It is interesting to plot the behaviour of the “cosmological function” Λ for our chosen modification. Since the non-classical branch (104) operates in the time interval $[-\tau_c, \tau_c]$, we have a strongly varying function $\Lambda(\tau)$, greatly deviating from the classical value Λ_0 in this region. This can be clearly seen in the plot of Fig. 3. The limit $\Lambda/\Lambda_0 \rightarrow 1$ as $|\tau| \rightarrow \infty$ is invisible in this plot due to resolution.

6.5 Modified $\Lambda_0 < 0$ case

In the case of GR, solution that we obtained in the time interval $(-\tau_c, \tau_c)$ was a realisation in the theory with $\Lambda_0 < 0$. In the modified gravity theory described above, we have obtained asymptotically De Sitter solutions with $\Lambda_0 > 0$ in the regions (τ_c, ∞) and $(-\infty, -\tau_c)$ and have shown that they are smoothly connected through a region of strong modified gravity in the interval $(-\tau_c, \tau_c)$. At a technical level, the strongly modified solution in the region $(-\tau_c, \tau_c)$ resulted from taking the negative root (104) of the quadratic equation (100) for $\Lambda(\tau)$.

In this subsection, we analyse the other possible solution. Specifically, in $(-\tau_c, \tau_c)$ we choose the solution of the quadratic equation that has the general-relativistic limit as $\alpha \rightarrow 0$. This is the solution that far from the would-be singularity behaves as that in GR with $\Lambda_0 < 0$. After that, we extend the solution beyond the interval $(-\tau_c, \tau_c)$ by taking the root (104) of the equation for $\Lambda(\tau)$. In this way, we obtain regions of strongly modified gravity in the intervals (τ_c, ∞) and $(-\infty, -\tau_c)$, connected by an almost GR solution with $\Lambda_0 < 0$ in the interval $(-\tau_c, \tau_c)$.

In order that the quadratic equation (100) always have solution for negative Λ_0 , we now should require that $\alpha < 0$. The expression under the square root in (101) and (104) will then always be positive. As $|\tau| \rightarrow \infty$, the function $x(\tau) \sim 2/9\tau^2$, so that $\Lambda_-(\tau) \sim -2/\alpha x \sim -9\tau^2/\alpha$. We thus have a strongly varying “cosmological function” in the regions (τ_c, ∞) and $(-\infty, -\tau_c)$.

It is not hard to repeat the analysis at $\tau = \pm\tau_c$ and see that there is no singularity in this solution. However, at some point $\tau_+ \in (\tau_c, \tau_1)$, the variable X_1 will now turn to zero and change sign. Repeating the analysis of Section 6.4, we conclude that, at this point, we will encounter singularity in metric (65), with a signature change from $(-, +, +, +)$ to $(-, +, -, -)$. A similar event will take place at another point $\tau_- \in (\tau_3, \tau_c)$, where it will be X_3 that will pass through zero and change sign. The signature of the metric right below τ_- will then become $(-, -, -, +)$. Furthermore, if $\tau_2 \neq 0$, then, depending on its sign, the variable X_2 will also pass through zero and change sign either at some $\tau > \tau_1$ (for $\tau_2 > 0$) or at some $\tau < \tau_3$ (for $\tau_2 < 0$). The signature of the metric will change accordingly at this point. Asymptotically, by calculating the limits of solution (97) as $|\tau| \rightarrow \infty$, we obtain

$$X_i \rightarrow \tau_i + \mathcal{O}(1/\tau), \quad (106)$$

The behaviour of X_i for $(\tau_1, \tau_2, \tau_3) = (1, 0, -1)$ is shown in Fig. 4.

In this case, the regions $\tau \rightarrow \pm\infty$ are singular. Indeed, integrating (52) using (106), and assuming all τ_i to be nonzero, we see that, for large times, $h_i \propto e^{\tau_i \tau}$. Thus, at least one of the connection (and metric) components exponentially blows up, while another exponentially shrinks to zero. This can be regarded as a true singularity of the solution, but “delayed” by the modification to occur at later times. The signature of metric (65) in the asymptotic regions is $(-, -, -, +)$ if $\tau_2 > 0$, and $(-, +, -, -)$ if $\tau_2 < 0$. If $\tau_2 = 0$, then this metric has the first signature in the region $\tau < \tau_-$, and the second signature in the region $\tau > \tau_+$.

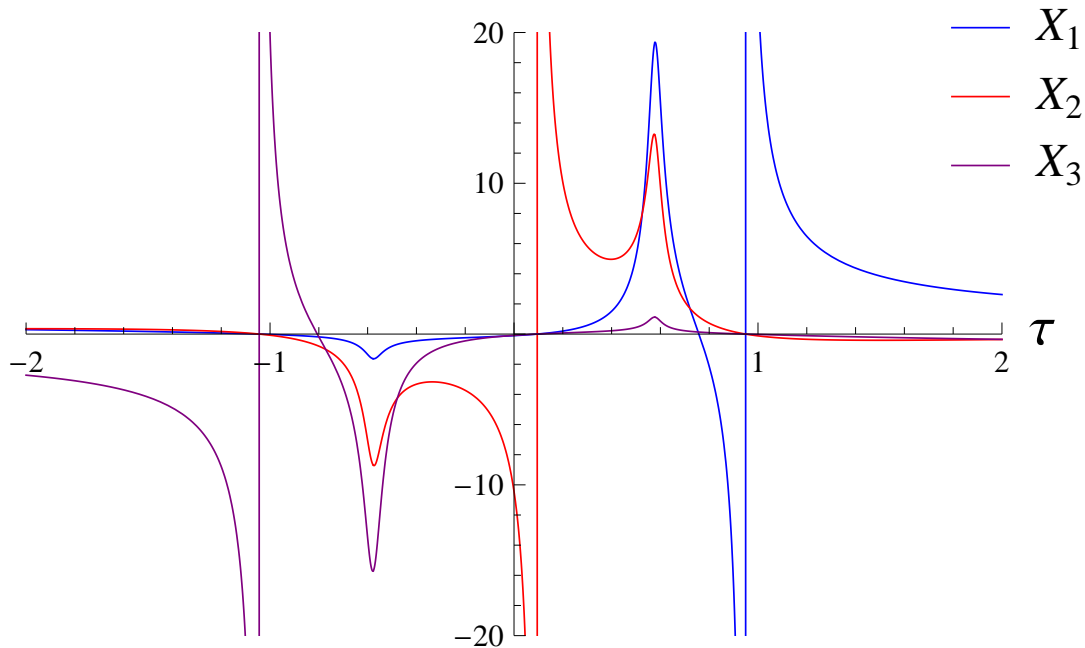


Figure 4: Behaviour of the variables X_i in the modified theory (16) with $\Lambda_0 < 0$ and $\alpha\Lambda_0 = 5 \times 10^{-3}$.

7 Discussion

In this paper, we reviewed the description of the family of chiral modified four-dimensional gravity theories. We used the language of the pure-connection formulation, in which the modification is particularly simple to describe. The idea is simply to consider more general functions of the curvature of the connection than the one that produces the GR dynamics. Such a modification is guaranteed to not lead to higher-order field equations. It may, however, result in the appearance of new propagating degrees of freedom. The chiral family of modified gravity theories considered in this work are known to have degrees of freedom like those present in GR — they continue to propagate just two polarizations of the graviton. Moreover, the argument of [3] based on scattering amplitudes shows that this is the only family of modified vacuum (i.e., not coupled to any matter) gravity theories in four spacetime dimensions that do not increase the number of propagating degrees of freedom.

On the technical side, we have introduced a new parametrisation of this family of theories. Previous studies of these chiral modified theories have used either the first-order BF-type description, as in [4, 5, 6, 7], or the pure-connection description of [8, 9]. The pure-connection formalism is much easier to deal with in practice, as the number of field components that one needs to consider is much smaller. However, it is not free from drawbacks. One of them is the impossibility to deal with the $\Lambda = 0$ sector of GR. Another

drawback is the presence of the square root in Lagrangian (32), with its several possible branches. Also, the field configurations where the matrix X^{ij} is degenerate (across some surfaces, for example) are problematic in the pure-connection formulation, leading to formal singularities of 0/0 type in the Euler–Lagrange equations. Finally, it is very difficult to control deviations from GR in the pure-connection formulation. Indeed, it is surprisingly difficult to describe, e.g., modification (16) in the pure-connection language.

All the mentioned drawbacks disappear if one allows oneself to keep, in addition to the connection, a matrix M^{ij} of auxiliary fields. One can then easily treat GR with zero cosmological constant, as we show in the Appendix on the example of the Bianchi I cosmology. There is no more need to take the matrix square root X^{ij} in obtaining the GR solution. Thus, configurations where one of the eigenvalues of the matrix X^{ij} turns to zero are no longer a problem, as we explicitly saw in the analysis of the GR solution in Section 5. It is also easy to describe controlled modifications of GR of the type (16), as we have seen in Section 6. All in all, we feel that “mixed” parametrisation introduced in this paper is superior to the pure-connection one, in the sense of keeping all its advantages (it is still a connection rather than metric formulation), while avoiding its drawbacks. This formulation of 4D gravity theories deserves to be studied in more detail, and we hope to return to it in future publications.

The family of modifications of four-dimensional General Relativity that was described in this paper is chiral. As a result, it is, in fact, a modification of *complexified* GR. It is not difficult to state the reality conditions leading to Riemannian and split signatures in the modified theories. However, no reality condition appropriate for the physical Lorentzian signature is yet known, at least not in full generality. So, as things stand, it is not clear whether the modified gravity theories described here admit physical interpretation.

Nevertheless, there are situations where the chiral character of the theory causes no difficulty. In this paper, we have studied one such situation. In Bianchi I spacetimes, the self-dual part of the Weyl curvature is real (even in Lorentzian signature). As a result, the modifications, described here, that effectively introduce the powers of the self-dual part of the Weyl curvature to the Lagrangian do not make the Lagrangian complex. Because of this, the metric arising in these modified theories can be taken to be real. In this paper, we obtained the behaviour of such real metrics in the Bianchi I setup by solving the evolution equations for the connection components and reading the metric from the connection.

Our main finding is that a natural one-parameter family of modifications (16), with a specific choice of the sign of the parameter α controlling the modification, resolves the Kasner singularity of Bianchi I spacetimes. In the most interesting case of modified gravity with a positive cosmological constant, we obtain a solution connecting two asymptotically De Sitter regions. This solution, depicted in Fig. 2, avoids the two would-be Kasner singularities of GR, remaining finite in this region. Between the two would-be singularities, there lies a region of strong modification of gravity. Inside this region, there are two moments of time where the metric becomes singular, while the basic fields of the theory (the connection components) remain finite. This behaviour is similar to what was observed in the case of modified black-hole solution in [5]. This type of metric singularity in the absence of any singularity in the basic fields appears to be generic to the modifier theories

under investigation.

The fact that modified gravity theories of this kind avoid so nicely the singularities of GR solutions suggests that they are more than just mathematical curiosities. At the same time, as we already noted, their physical interpretation is not clear at present. In a more general setting, it is no longer possible to require that the metric as defined by the connection via (33) is real Lorentzian, as this is simply incompatible with the dynamics. It is not yet clear how this inherent complexity of the theory should be treated. There might exist some choice of reality conditions that reduces the dynamics to a half-dimensional slice through the phase space with real-valued Hamiltonian and symplectic form on this slice. If this is the case, such a real slice would provide the sought interpretation.

A related problem is that of coupling to matter. The fundamental field of our gravity theories is a connection, to which the matter fields should couple directly. Such coupling in the case of GR can be obtained by the same trick of integrating out the metric. Indeed, in the first-order formalism, all matter Lagrangians are algebraic in the metric (i.e., do not contain derivatives of $g_{\mu\nu}$). Thus, in principle at least, the metric can still be integrated out to produce a connection-plus-matter formulation. In practice, however, this may be difficult. Further, it is not clear how to modify such gravity-plus-matter theories. Alternatively, one can start with a more general family of gauge theories as described in [18]. These describe gravity as well as matter fields. So, some type of matter couplings can be obtained in this way. But the problem of finding the correct reality conditions still remains in any setting — since matter fields couple to a complex-valued connection, some reality conditions are required to make sense of the arising dynamics.

We end this paper by noting that the problem of reality conditions is the most pressing one as far as classical theory is concerned. We hope that the results presented here, even though not immediately concerned with this problem, will serve as a step towards developing a physical interpretation of the modified gravity theories under consideration.

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Appendix: The sector of interest

In this Appendix, we derive the diagonal ansatz used in the main text from more general considerations.

We are interested in a general spatially homogeneous but anisotropic sector of the theory. This is described by an $SO(3)$ connection with components A^i that are functions

of the time coordinate τ only. A general such connection is of the form

$$A^i = a^{ij} dx^j + b^i d\tau, \quad (107)$$

where a^{ij} and b^i are (complex-valued in general) functions of the time coordinate τ . Performing a time-dependent $\text{SO}(3, \mathbb{C})$ transformation, one can always make the matrix a^{ij} symmetric. After that, using τ -dependent spatial translations, one can always set the b -components to zero.

Thus, gauge-fixing the spatial diffeomorphisms and the gauge rotations, we are led to consider the following sector of the theory:

$$A^i = a^{ij} dx^j, \quad (108)$$

where a^{ij} is a symmetric matrix of arbitrary (complex-valued) functions of the time coordinate only. In more mathematical language, the time evolution of our system is a trajectory on the homogeneous group manifold

$$\text{GL}(3, \mathbb{C})/\text{SO}(3, \mathbb{C}), \quad (109)$$

parametrised by symmetric complex-valued 3×3 matrices. We have not yet imposed any reality conditions, this will be done below.

Field equations

The field equations in the pure-connection theory are

$$d_A \left(\frac{\partial f}{\partial X^{ij}} \right) \wedge F^j = 0, \quad (110)$$

where the matrix X^{ij} is defined as in (23) and, in view of the homogeneity of the function $f(\cdot)$, it does not matter precisely which volume form is chosen in (23).

For the connection (108) we have:

$$F^i = \dot{a}^{ij} d\tau \wedge dx^j + \frac{1}{2} (a^{-1})^{ij} \det a \epsilon^{jkl} dx^k \wedge dx^l, \quad (111)$$

where we have assumed the matrix a^{ij} to be invertible,¹ and the dot denotes the derivative with respect to τ . Correspondingly, the matrix of the wedge products of the curvature is given by:

$$F^i \wedge F^j = 2\dot{a}^{(i|k|} (a^{-1})^{j)k} \det a d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3, \quad (112)$$

so that

$$X^{ij} \propto \dot{a}^{(i|k|} (a^{-1})^{j)k}, \quad (113)$$

where the proportionality means equality modulo an arbitrary function of time.

¹We define the inverse matrix by the property $a^{ik} (a^{-1})^{jk} = \delta^{ij}$.

Now we can write the field equations. It is not hard to check that the $dx^i \wedge dx^j \wedge dx^k$ component of equations (110) holds automatically, in view of the symmetry of the matrix of first derivatives of the function $f(X)$. Thus, we only need to consider the $\epsilon^{ijk} d\tau \wedge dx^j \wedge dx^k$ part. This part reads:

$$\left(\frac{\partial f}{\partial X^{ij}} \right) \cdot (a^{-1})^{jk} \det a - \epsilon^{ipl} \epsilon^{mnk} a^{pm} \dot{a}^{jn} \frac{\partial f}{\partial X^{lj}} - \epsilon^{jpl} \epsilon^{mnk} a^{pm} \dot{a}^{jn} \frac{\partial f}{\partial X^{li}} = 0. \quad (114)$$

It is convenient to multiply this equation by a^{qk} and divide by $\det a$. After some simple algebra, we get

$$\left(\frac{\partial f}{\partial X^{ij}} \right) + \frac{\partial f}{\partial X^{ij}} \text{Tr } X + \frac{\partial f}{\partial X^{jk}} (X + Y)^{ki} - \frac{\partial f}{\partial X^{ik}} (X + Y)^{jk} = \delta^{ij} \text{Tr} \left(\frac{\partial f}{\partial X} X \right). \quad (115)$$

Here, we have introduced the notation

$$\dot{a}^{ik} (a^{-1})^{jk} = X^{ij} + Y^{ij}, \quad (116)$$

where X and Y are the symmetric and antisymmetric parts, respectively. Now, in view of the gauge invariance of the function $f(X)$, we have:

$$\frac{\partial f}{\partial X^{jk}} X^{ki} - \frac{\partial f}{\partial X^{ik}} X^{kj} = 0, \quad (117)$$

and only the Y -part survives in the third and fourth terms in (115). We can also use the homogeneity of $f(X)$ that implies

$$\text{Tr} \left(\frac{\partial f}{\partial X} X \right) = f. \quad (118)$$

Eventually, we get the differential equation for X^{ij} :

$$\left(\frac{\partial f}{\partial X^{ij}} \right) + \frac{\partial f}{\partial X^{ij}} \text{Tr } X + \frac{\partial f}{\partial X^{ik}} Y^{kj} + \frac{\partial f}{\partial X^{jk}} Y^{ki} = \delta^{ij} f(X). \quad (119)$$

Note that both the left-hand and right-hand side are explicitly symmetric in ij , as they should be.

Reality conditions

Let us now impose the reality conditions (34). For the sector of interest, they read

$$\dot{a}^{ik} (\bar{a}^{-1})^{jk} \det \bar{a} + (\dot{\bar{a}})^{jk} (a^{-1})^{ik} \det a = 0. \quad (120)$$

We will not attempt at finding the most general solution of this equation, considering instead a particular ansatz sufficient for our purposes. Thus, we require that a^{ij} are all purely imaginary:

$$a^{ij} = ih^{ij}, \quad h^{ij} \in \mathbb{R}. \quad (121)$$

Condition (120) then boils down to

$$(\dot{h}h^{-1})^{[ij]} = 0. \quad (122)$$

This, in particular, implies that $Y^{ij} = 0$ in (119).

Condition (122) can be stated as requiring that the matrices \dot{h} and h commute at all times. Let us consider some initial moment of time. Then we can simultaneously diagonalise both of these symmetric matrices by an orthogonal transformation. Then the evolution equations (119) can be seen to imply that if \dot{h} and h are diagonal at the initial moment of time, they will stay diagonal. So, without loss of generality, we can assume h^{ij} to be diagonal at all times:

$$h^{ij} = \text{diag}(h_1, h_2, h_3). \quad (123)$$

The evolution equations then take the following simple form:

$$\left(\frac{\partial f}{\partial X^{ij}} \right)' + \frac{\partial f}{\partial X^{ij}} \text{Tr } X = \delta^{ij} f(X), \quad (124)$$

$$X^{ij} = \text{diag}(X_1, X_2, X_3), \quad X_i = \frac{\dot{h}_i}{h_i}. \quad (125)$$

It can, moreover, be assumed that all matrices appearing here are diagonal.

System (124), (125) can be viewed as a system of second-order differential equations for the functions h_1 , h_2 , and h_3 . However, it is more convenient to view (124) as a system of first-order differential equations for the components of the (diagonal) symmetric matrix X^{ij} . Once these are found, the components of the connection can be found by integrating equation (125). The function $f(X)$ should be considered as given. In the main text, we study equations (124) using the parametrisation of the function $f(X)$ in terms of M , as described in Section 2.

Appendix: GR solution in the physical time

In this section, we find the GR solution of the Bianchi I model working in the physical time coordinate, see below. We give it here for completeness, as well as to stress the point that the time variable used in the main text simplifies it considerably.

For the function $f(X)$ given by (32), which corresponds to general relativity, equation (54) reduces to

$$\left(\frac{\sum_j y_j}{y_i} \right)' = \left(\sum_j y_j \right)^2 - \frac{\sum_j y_j}{y_i} \sum_j y_j^2, \quad (126)$$

where $y_j = \sqrt{X_j}$ is understood as some branch of the square root.

The physical time

It is clear from (65) that the choice of physical time corresponds to the following condition

$$\text{Tr } \sqrt{X} \det \sqrt{X} = \Lambda, \quad (127)$$

where Λ is, in fact, the cosmological constant, which, for definiteness, we assume to be positive. We shall denote the physical time by t .

Solution in the physical time

Let us choose to parametrise X_i as follows:

$$X_i = \frac{\epsilon_{ijk} H_j H_k}{2H_i}. \quad (128)$$

Condition (127) then takes the form familiar from the Bianchi I cosmology, in which H_i play the role of the Hubble parameters:

$$H_2 H_3 + H_3 H_1 + H_1 H_2 = \Lambda. \quad (129)$$

We then have

$$\sum_j y_j = (H_1 H_2 H_3)^{-1/2}, \quad \frac{\sum_j y_j}{y_i} = \frac{2}{\epsilon_{ijk} H_j H_k}. \quad (130)$$

Substituting these expressions into (126) and using constraint (129), we obtain the usual evolution equations for the Bianchi I cosmology:

$$\dot{H}_i + H_i^2 = \frac{1}{2} \epsilon_{ijk} H_j H_k. \quad (131)$$

Solution of these equations can be parametrised by angle θ and presented in the form

$$\begin{aligned} H_1 &= \frac{1}{3\sqrt{\Lambda}} \left[\coth \sqrt{\Lambda} t + \frac{2 \sin \theta}{\sinh \sqrt{\Lambda} t} \right], \\ H_2 &= \frac{1}{3\sqrt{\Lambda}} \left[\coth \sqrt{\Lambda} t + \frac{2 \sin \left(\theta + \frac{2\pi}{3} \right)}{\sinh \sqrt{\Lambda} t} \right], \\ H_3 &= \frac{1}{3\sqrt{\Lambda}} \left[\coth \sqrt{\Lambda} t + \frac{2 \sin \left(\theta - \frac{2\pi}{3} \right)}{\sinh \sqrt{\Lambda} t} \right]. \end{aligned} \quad (132)$$

For $\Lambda > 0$, it describes the Kasner regime as $t \rightarrow 0$ and proceeds to the De Sitter space asymptotically as $t \rightarrow \infty$. For $\Lambda < 0$, the solution is obtained by analytic continuation, and describes evolution between two Kasner-like singularities.

The above solution for H 's gives us the functions X_1 , X_2 , and X_3 . We then can integrate (52) and get the components of the connection. An equivalent, but simpler procedure is to look for h 's in the parametrisation

$$h_i = H_i a_i, \quad (133)$$

where a_i are some yet unknown functions. Then, if we write

$$\frac{H_2 H_3}{H_1} = \frac{(H_1 a_1)'}{H_1 a_1} = \frac{(H_2 H_3 - H_1^2) a_1 + H_1 \dot{a}_1}{H_1 a_1}, \quad (134)$$

where we have used one of equations (131), we see that

$$\frac{\dot{a}_i}{a_i} = H_i, \quad (135)$$

which allows one to solve for a_i and then get the functions h_i from (133). From (57), we also note that the functions a_i are precisely the scale factors of the metric, i.e.,

$$ds^2 = -dt^2 + \sum_i a_i^2 (dx^i)^2. \quad (136)$$

In the next section, we show how to obtain the GR solution in a simpler way by choosing a different time coordinate.

Appendix: GR with zero cosmological constant

In this Appendix, we find the solution of GR with zero cosmological constant using our mixed parametrisation. The purpose of this exercise is to see that the mixed parametrisation can be readily used to solve the equations of GR with $\Lambda = 0$, where no pure-connection formulation is available.

For the same Bianchi I ansatz, the evolution equations take the following form. First, there is equation (56), which in this case takes the form

$$(\Psi_i^{-1} h)' = \text{Tr} (\Psi^{-1} X) h = 0, \quad (137)$$

where the zero right-hand side follows from equation (41). We then get

$$\Psi_i = \sigma_i h, \quad (138)$$

where σ_i are nonzero integration constants.

Equation (41) and the tracelessness of Ψ then imply, respectively,

$$X_i = \zeta(\tau) \sigma_i^2, \quad \sum_i \sigma_i = 0, \quad (139)$$

where $\zeta(\tau)$ in the first equation is some function of time. The second equation entails a useful relation

$$\sum_i \sigma_i^4 = \sum_{i \neq j} \sigma_i^2 \sigma_j^2. \quad (140)$$

We look for solution of the theory in the form of power law:

$$h_i(\tau) = C_i \tau^{m_i}, \quad (141)$$

with some exponents m_i . With this ansatz for the solution, we have

$$X_i = \frac{\dot{h}_i}{h_i} = \frac{m_i}{\tau}. \quad (142)$$

Equations (139) give $\zeta(\tau) = C/\tau$, and imply the following relations between m_i and σ_i :

$$m_i = C \sigma_i^2. \quad (143)$$

Equation (140) then gives

$$\sum_i m_i^2 = \sum_{i \neq j} m_i m_j. \quad (144)$$

We now need to specify the time coordinate which, up to now, was quite arbitrary. Let us choose τ to be the physical time coordinate t , in which $N^2(t) \equiv 1$. Declaring the curvature two-forms to be self-dual, we get relations (60). With the above choice for the lapse function, this gives

$$a_1^2 = \frac{C_1^2}{m_2 m_3} t^{2(m_1+1)}, \quad \text{etc.} \quad (145)$$

On the other hand, according to (13), the metric volume form (61) should be a multiple of

$$\text{Tr}(B \wedge B) = \text{Tr}(\Psi^{-1} F \wedge \Psi^{-1} F) \propto \text{Tr}(\Psi^{-2} X) h \mathcal{V}_c \quad (146)$$

[see (51)]. This gives

$$\prod_i a_i \equiv \left| \frac{h}{\prod_i X_i} \right| \propto (ht)^{-1}, \quad (147)$$

which, in turn, implies $h(t) \propto t^{-2}$, and

$$\sum_i m_i = -2. \quad (148)$$

Comparing (145) to the Kasner metric (93) we see that we must identify

$$m_i = p_i - 1. \quad (149)$$

Then (148) takes the familiar form

$$\sum_i p_i = 1, \quad (150)$$

and (144) implies

$$\sum_i p_i^2 = 1. \quad (151)$$

We thus recover the Kasner solution.

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